International Journal of Theoretical Physics, Vol. 38, No. 1, 1999

Classical Limit in Fuzzy Set Models of Spin-1/2 Quantum Logics

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Received August 7, 1998

The classical limit of fuzzy set models of spin-1/2 quantum logics is obtained in the course of a "defuzzyfication" procedure. The conditions under which the limiting structure is a Boolean algebra are studied.

1. INTRODUCTION

The idea that the classical description of physical phenomena can be obtained from the quantum description by some kind of a limit procedure belongs to the folklore of physics. This idea can be easily illustrated within the standard formulation of quantum mechanics where, roughly speaking, the classical description is obtained when the Planck constant converges to zero and quantum numbers tend to infinity.

However, within the quantum logic theory, although quantum logics modeled by orthomodular σ-orthocomplete partially ordered sets are straightforward generalizations of Boolean σ-algebras, which in turn are structures characteristic of classical physics, the possibility of passing from a general quantum logic to a Boolean algebra in the course of some limit procedure has not yet been investigated.

Actually, traditional, i.e., order-theoretic, models of quantum logics leave hardly any room for the desired limit procedure: Quantum logics modeled in such a way are "too stiff" structures to undergo "infinitesimal changes" required by a limit process. Such a possibility is left open within the fuzzy set approach to quantum logic theory developed by Pykacz (1987, 1992,

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0020-7748/99/0100-0387$16.00/0

1994, 1998) in which quantum logics are represented by families of fuzzy sets. This possibility follows from the fact that Boolean algebras of classical physics consist of traditional (in fuzzy set terminology, *crisp*) sets, which in turn can be treated as "limiting cases" of fuzzy sets that, in the process of "defuzzyfication," become "less and less fuzzy." There are many ways in which fuzzy sets can be made "less fuzzy" described in the vast literature on fuzzy sets. In this paper we utilize a very specific one which is motivated by physical considerations. It is very similar to the procedure studied in a physical context by Aerts *et al.* (1992). In that paper the procedure of "defuzzyfication" was applied to fuzzy sets that describe properties of spin-1/2 quantum particles and it was shown that crisp sets are obtained in the classical limit, as expected. However, the scope of that paper was restricted to studying how a single fuzzy set which represents only one property of a physical system changes when a situation becomes "more classical." The aim of the present paper is to study the classical limit of the *whole* quantum logic modeled by a suitable family of fuzzy sets. In particular we show that the "defuzzyfication" procedure alone, although it yields a family of crisp sets, does not suffice to endow this family with a structure of a Boolean algebra and we investigate what should be added to this procedure to finally get this desired structure.

The paper is organized as follows: Section 2 is devoted to the general outline of the fuzzy set approach to quantum logics. In Section 3 the behavior of quantum spin-1/2 particles is described within the "hidden measurement" model and we show that this model in a natural way gives rise to a fuzzy set representation of properties of spin-1/2 particles. In Section 4 we construct a fuzzy-set model for two-dimensional Hilbertian quantum logic which consists of fuzzy subsets of a Poincaré sphere, and a "defuzzyfication" procedure is applied to this specific fuzzy set quantum logic.

2. FUZZY SET MODELS OF QUANTUM LOGICS

By a *quantum logic* (or simply a *logic*, since these structures are typical to not only quantum physical systems) we mean in the present paper an orthocomplemented, σ-orthocomplete, orthomodular poset, i.e., a partially ordered set L which contains the smallest element O and the greatest element I, in which the orthocomplementation map $\perp: L \rightarrow L$ satisfying the following conditions exists:

(a) $(a^{\perp})^{\perp} = a$.

(b) If $a \leq b$, then $b^{\perp} \leq a^{\perp}$.

(c) The greatest lower bound (*meet*) $a \wedge a^{\perp}$ and the least upper bound (*join*) $a \vee a^{\perp}$ with respect to the given partial order exist in L for any $a \in L$ and $a \wedge a^{\perp} = O$, $a \vee a^{\perp} = I$.

Moreover, the σ-orthocompleteness condition holds:

(d) If $a_i \leq a_j^\perp$ for $i \neq j$ (such elements are called *orthogonal*), then the join $\bigvee_i a_i$ exists in L.

And so does the orthomodular identity:

(e) If $a \leq b$, then $b = a \vee (a^\perp \wedge b) = a \vee (a \wedge b^\perp)^\perp$.

Elements $a, b \in L$ are called *compatible* iff there exist in L pairwise orthogonal elements a_1, b_1, and c such that $a = a_1 \vee c$ and $b = b_1 \vee c$.

According to the standard interpretation, to any physical system we can associate a logic L consisting of elementary ("yes–no") propositions about this system or, equivalently, properties of this system. *Probability measures* on the logic L, i.e., mappings s: $L \to [0, 1]$ such that

(i) $s(I) = 1$
(ii) $s(\vee_i a_i) = \Sigma_i s(a_i)$ for any sequence consisting of pairwise orthogonal elements of L

represent states of a physical system and therefore are usually themselves called *states on L*. A number $s(a) \in [0, 1]$ is usually interpreted as a probability of obtaining the result "yes" in an experiment designed to check a proposition a when a physical system is in a state represented by s (equivalently: the probability of finding that a system in the state s has the property a). A set of states S is called *ordering* (or *full, order determining*) iff $s(a) \leq s(b)$ for all $s \in S$ implies $a \leq b$.

Two most standard examples of logics of physical systems are the following:

C. Logic of a classical statistical system is a Boolean algebra $\mathcal{B}(\Gamma)$ consisting of Borel subsets of a phase space Γ of a system. All elements of $\mathcal{B}(\Gamma)$ are compatible, as happens in any Boolean algebra. States of a physical system are represented by usual Kolmogorovian probability measures on $\mathcal{B}(\Gamma)$, pure states being represented by Dirac measures concentrated on one-point subsets of Γ or, equivalently, by points of Γ on which these measures are concentrated.

Q. Logic of a quantum system described with the aid of a Hilbert space $\mathcal{H}$ is an orthomodular lattice $L(\mathcal{H})$ of closed subspaces of $\mathcal{H}$ or, equivalently, of orthogonal projections onto these closed subspaces. Only commuting projections are compatible. States are represented by density operators, pure states being represented by projections onto one-dimensional subspaces or, equivalently, by unit vectors which determine these subspaces.

In both cases the set of all states on a logic is ordering.

Let $U \neq \emptyset$ be a fixed set called a *universe*. According to Zadeh (1965), a *fuzzy set* $\mathcal{A}$ in U is defined by its membership function $\mu_{\mathcal{A}}: U \to [0, 1]$ in such a way that for any $x \in U$ the number $\mu_{\mathcal{A}}(x) \in [0, 1]$ represents the *degree of membership* of the point x to the fuzzy set $\mathcal{A}$. Characteristic functions of traditional crisp sets are of course special cases of membership functions; therefore, fuzzy sets are generalizations of traditional sets. Many authors identify fuzzy sets with their membership functions and write $\mathcal{A}(x)$ instead of $\mu_{\mathcal{A}}(x)$. This convention is adopted throughout the rest of this paper. Another convention adopted in this paper consists in denoting all fuzzy sets (which, in particular, can be also crisp) by capital script letters and reserving capital roman characters for traditional sets, i.e., such crisp sets that are never fuzzy.

Already in his first paper on fuzzy sets Zadeh proposed the following expressions for various relations between fuzzy sets and operations on them:

$$\mathcal{A} = \mathcal{B} \text{ iff } \mathcal{A}(x) = \mathcal{B}(x) \tag{1}$$

$$\mathcal{A} \subseteq \mathcal{B} \text{ iff } \mathcal{A}(x) \leq \mathcal{B}(x) \tag{2}$$

$$\mathcal{B} = \mathcal{A}' \text{ iff } \mathcal{B}(x) = 1 - \mathcal{A}(x) \tag{3}$$

$$\mathcal{C} = \mathcal{A} \cap \mathcal{B} \text{ iff } \mathcal{C}(x) = \min[\mathcal{A}(x), \mathcal{B}(x)] \tag{4}$$

$$\mathcal{C} = \mathcal{A} \cup \mathcal{B} \text{ iff } \mathcal{C}(x) = \max[\mathcal{A}(x), \mathcal{B}(x)] \tag{5}$$

(the right-hand sides of the above equivalences hold for all $x \in U$).

However, Zadeh's "min" (4) and "max" (5) expressions are not the only possible expressions for intersection and union of fuzzy sets. Actually, there is an infinite variety of them, some of them being more and some less natural or plausible. Another pair of intersection and union [also forming a De Morgan triplet together with the standard fuzzy set complement (3)] was introduced to the fuzzy set theory by Giles (1976):

$$(\mathcal{A} \sqcap \mathcal{B})(x) = \max[\mathcal{A}(x) + \mathcal{B}(x) - 1, 0] \tag{6}$$

$$(\mathcal{A} \sqcup \mathcal{B})(x) = \min[\mathcal{A}(x) + \mathcal{B}(x), 1] \tag{7}$$

Giles called these operations on fuzzy sets *bold* operations and they appear in the literature also under the names of *Giles, bounded, truncated*, or *Lukasiewicz* operations. It occurs (Mesiar, 1994; Pykacz, 1997) that only these (or slightly more general, but "isomorphic" to them) operations on fuzzy sets can be used for building fuzzy set models of quantum logics.

The possibility of representing quantum logics by suitable families of fuzzy sets is secured by the following theorem (Pykacz, 1994):

Theorem 1. Any quantum logic L with an ordering set of states S can be represented in the form of a family $\mathcal{L}(S)$ of fuzzy subsets of S satisfying the following conditions:

(a) $\mathcal{L}(S)$ contains the empty set $\emptyset$, i.e., such set that $\emptyset(s) = 0$ for all $s \in S$.

(b) $\mathcal{L}(S)$ is closed under the complementation (3).

(c) $\mathcal{L}(S)$ is closed under countable Giles unions of pairwise *weakly disjoint* sets, i.e., if $\mathcal{A}_i \sqcap \mathcal{A}_j = \emptyset$ for $i \neq j$, then $\sqcup_i \mathcal{A}_i \in \mathcal{L}(S)$.

(d) If $\mathcal{A} \sqcap \mathcal{A} = \emptyset$, then $\mathcal{A} = \emptyset$.

Conversely, any family of fuzzy subsets of an arbitrary universe U satisfying conditions (a)–(d) is a quantum logic partially ordered by the inclusion of fuzzy sets (2), with the fuzzy set complementation (3) as orthocomplementation, orthogonality of elements coinciding with their weak disjointedness, and an ordering set of states generated by points of the universe U according to the formula

$$s_x(\mathcal{A}) = \mathcal{A}(x) \tag{8}$$

It should be stressed that the assumption that a logic should possess an ordering set of states is physically well justified. Actually, it is unavoidable if we agree that any knowledge about relations between various elements of a logic should be obtained by performing experiments on (copies of) a physical system prepared in various states.

The interpretation of fuzzy subsets of the set of states S that belong to the family $\mathcal{L}(\mathcal{S})$ mentioned in Theorem 1 is straightforward: a fuzzy subset $\mathcal{A}$ of S representing an element a of a logic L, therefore, also a property of a physical system, collects all states for which this property holds. However, according to the very idea of fuzzy sets, degrees of membership of various states to the set $\mathcal{A}$ represent various "extents" to which this property holds when a physical system is in various states. To illustrate this statement let us consider linearly polarized photons and a property a of *passing through a linear polarizer oriented under the angle* α *to the direction of polarization of incoming photons*. Although all photons are in the same pure state, we only know that for each photon there is a probability $\cos^2 \alpha$ that it will pass through the polarizer and a probability $\sin^2 \alpha$ that it will be stopped. Does this mean that some photons in the beam possess the property a and others do not possess it even before they reach the polarizer? Such a position could be taken only by advocates of "orthodox" hidden variable theories, which are based on the assumption that all properties of quantum objects are predetermined. The adherent of the orthodox Copenhagen interpretation would simply say that it is meaningless to talk about any property of a quantum object before a suitable experiment (in our case a trial of passing through the polarizer) is completed. However, both these positions are based on the classical two-valued logic, which is also the basis of traditional set theory. Fuzzy set theory (or, equivalently, infinite-valued logic) allows a third possibility, applicable to both quantum mechanics and "hidden measurement" approach: we can legitimately say that each of our photons, before it reaches the polarizer, both *has the property a to the degree* $\cos^2 \alpha$ and simultaneously *has not the property a to the degree* $\sin^2 \alpha$. Therefore, instead of repeating the Peres' (1978) statement, "*unperformed experiments have no results*," we would rather say, "*unperformed experiments have simultaneously all their possible results, each of them to the extent allowed by suitable quantum mechanical calculations*."

3. HIDDEN MEASUREMENTS MODEL OF SPIN-1/2 PARTICLES AND ITS FUZZY SET DESCRIPTION

In this section we describe a simple "classical-stochastic" machine that yields a faithful mathematical description of a spin-1/2 particle. It belongs to a family of "hidden measurements" models of quantum systems intensively studied in Brussels since 1983 (see, e.g., Aerts, 1983, 1986, 1987). In quantum theory a spin-1/2 particle is described with the aid of a two-dimensional complex Hilbert space. Pure states of the entity are represented by rays in that Hilbert space. We want to study the system during a continuous transition from a quantum entity toward a classical entity. It can be shown (Aerts and Durt, 1994a, b) that in these intermediate situations the Hilbert space structure of the state space will be lost. Therefore we have to use another representation of the state space of the entity. Following Mielnik (1968), we shall map the unit vectors of the two-dimensional complex Hilbert space on the surface of a unit sphere in three dimensions, which in this case is usually called the Poincaré sphere. In this procedure we make use of the connection between the measurement direction of a Stern–Gerlach experiment in three-dimensional space and the operator representing the spin observable and acting on the Hilbert space. We use the following mapping:

$$S_U: \quad u = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \to S_u = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix} \tag{9}$$

which maps a unit vector u on the spin-1/2 operator S_u. This self-adjoint spin operator has two orthogonal eigenvectors which form a basis for the Hilbert space $\mathbb{C}^2$, namely

$$s_{u_+} = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\varphi/2} \\ \sin\frac{\theta}{2}e^{i\varphi/2} \end{pmatrix}, \qquad s_{u_-} = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\varphi/2} \\ \cos\frac{\theta}{2}e^{i\varphi/2} \end{pmatrix} \tag{10}$$

with eigenvalues $+1/2$ and $-1/2$, respectively. The physical meaning of these eigenvectors is that if the entity is in a state s_u+; we will find with certainty the outcome $+1/2$ for the spin measurement along the direction defined by u. We now let the direction defined by u correspond with this eigenvector and simply say that the particle in the state s_u+ is represented on the Poincaré sphere by the point u. In short, we make a point u of the surface of the Poincaré sphere correspond with an eigenvector s_u+ in $\mathbb{C}^2$. This correspondence between the set of pure states of quantum spin-1/2 particles and the elements of the surface of the Poincaré sphere is one to one (Aerts and D'Hooghe, 1996) and it allows us to study a "classical-stochastic" entity with a set of states given by the points on the Poincaré sphere which therefore reproduces all numerical results of quantum spin-1/2 measurements. It allows also to study a set of experiments parametrized by a parameter δ, which indicates the amount of "stochasticity" of the experiment and makes it possible to pass to the "classical limit" when the amount of "stochasticity" tends to zero.

The entity consists of a point particle, and the states of the entity are given by the points p_θ of the Poincaré sphere. The "classical-stochastic" model for a measurement of the spin of the particle along the direction u that we shall use goes as follows (see Fig. 1): we place an elastic string of length 2 between u and $-u$. Then the particle falls orthogonally onto the elastic and sticks to it at a point that we shall denote by p'_θ. Then the elastic breaks at a random point with uniform probability. If it breaks between p'_θ and $-u$, the particle goes toward u and stays there. The measurement reveals the outcome "spin up." If the elastic breaks between p'_θ and u, the particle moves toward $-u$ and stays there, and the measurement reveals the outcome "spin down." If the string breaks at exactly the point where the particle is attached, then we assume that in 50% of the cases the particle moves toward u, and the outcome is "spin up," and in the other 50% of the cases the particle moves toward $-u$, and the outcome is "spin down." We remark that these events have a measure zero, and we can see them as an analogue of the "unstable equilibrium" of classical mechanics.

The probabilities of the respective outcomes are as follows. We shall use the notation θ to denote the angle between the initial state of the entity and the direction u of the measurement device. The probability for

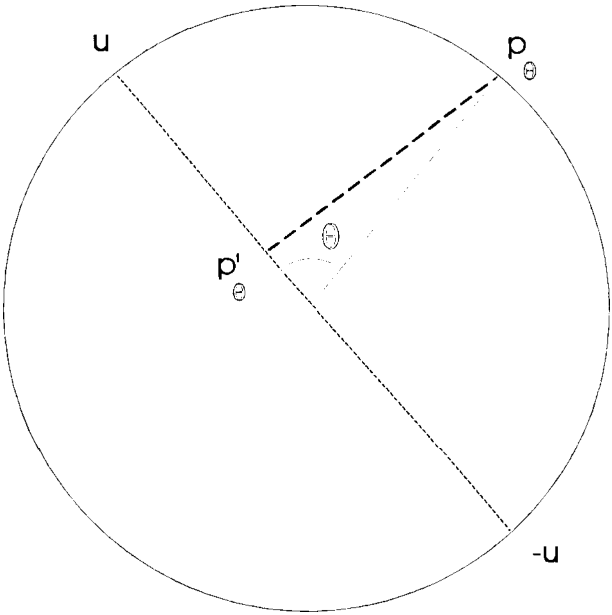

Fig. 1. The "classical-stochastic" model for a measurement of the spin of a particle along the direction u in its quantum limit $\delta = 1$.

"spin up" is given by the length of the elastic between the projection point p'_θ and the point u, normalized by the total length of the elastic. This is $(1 - \cos\theta)/2 = \cos^2(\theta/2)$. Similarly we can calculate the probability for the "spin down" outcome as $(1 + \cos\theta)/2 = \sin^2(\theta/2)$. These probabilities coincide with the quantum probabilities for a spin measurement of a spin-1/2 particle. These probabilities can be given a geometrical interpretation which makes the transition toward the fuzzy set approach very natural. Since the probabilities are only determined by the angle between the initial state and the measurement direction, we can use one great circle with axis $[u, -u]$ for the representation of the set of states instead of the whole Poincaré sphere. We then take a cylinder with this circle as ground circle and height 1 (see Fig. 2).

Next we take a plane which intersects this cylinder along an ellipse such that one of the endpoints of the long axis of the ellipse is $-u$ and the other lies in the upper plane of the cylinder straight above u. When we unfold the surface of the cylinder, the ground circle becomes a straight-line interval of the length 2π and the intersection line of the ellipse with the cylinder becomes a curve which reaches in every point p_θ of the straight line value $\cos^2(\theta/2)$. Therefore, the height of the intersection line

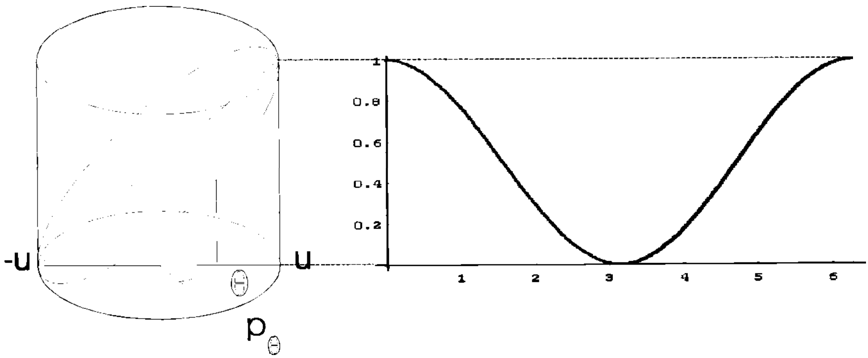

Fig. 2. Geometrical interpretation of the transition probabilities for a quantum spin-1/2 particle.

at each point p_θ coincides with the probability that for a quantum spin-1/2 particle in the pure state represented by the point p_θ the measurement will yield the outcome "spin up."

We now introduce a parameter in this model, $\delta \in [0, 1]$, which describes the "shielding" of the elastic. First let us consider the case $\delta \in (0, 1]$, which is drawn in Fig. 3.

Around and parallel with the elastic we place two semiopen cylindrical tubes, both of length $1 - \delta$, one with u as the center of its ground plane,

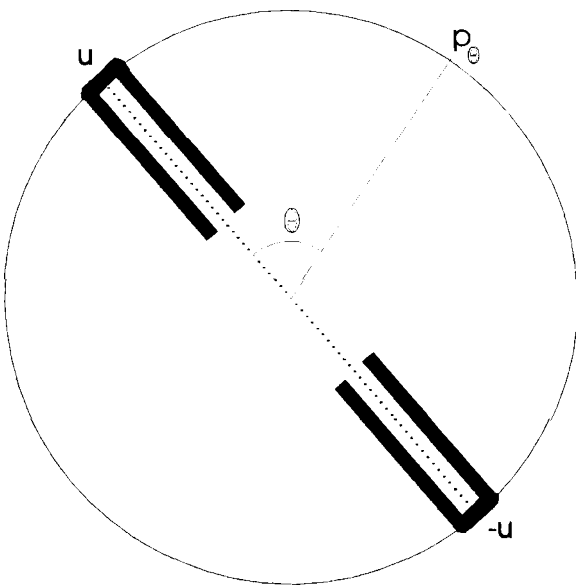

Fig. 3. The model with a shielding of the elastic string, parametrized by δ.

the other one with $-u$ as the center of its ground plane. Then the particle falls orthogonally onto the measurement direction $[u, -u]$. If it falls onto one of the two shielding tubes, it slides toward the center point of that cylinder (thus either u or $-u$) and sticks to the endpoint, and the experiment yields the corresponding outcome. For instance, if the point particle falls onto the tube with center point u, it slides toward u, stays there, and the experiment yields the outcome "spin up." If the particle falls on the elastic between the two tubes, it sticks to the elastic at that orthogonal projection point p'_θ and the further process follows as in the "pure quantum" case described above. In the case $\delta = 0$ we place two shielding tubes of length 1 in the same manner as before around the elastic, but in such a way that there is one point in the middle of the elastic which is not shielded by either of the tubes. The measurement is done in the same manner as before: if the orthogonal projection of the state lies on the upper tube (with center u), then the particle moves toward u, stays there, and gives outcome "spin up." If the orthogonal projection of the state lies on the lower tube (with center $-u$), then the particle moves toward $-u$, stays there, and gives outcome "spin down." If the projection point of the initial state lies exactly in the middle of the elastic (not shielded by either of the two tubes), then we say that in 50% of the cases it moves toward u and "spin up" is measured, and in 50% of the cases it moves toward $-u$ and "spin down" is measured. Again we notice that this event has measure zero, and that we can see it as an analogue of the "unstable equilibrium" of classical mechanics.

The probabilities of the respective outcomes are as follows. We shall use again the notation θ to denote the angle between the initial state of the entity and the direction u of the measurement device, and the notation θ_δ for the angle for which $\cos \theta_\delta = \delta$. For $\delta \neq 0$, the states which make an angle $\theta \in [-\theta_\delta, \theta_\delta]$ will always give the outcome "spin up." If the state of the entity makes an angle $\theta \in [\pi - \theta_\delta, \pi + \theta_\delta]$ with the measurement direction u, then we always get the outcome "spin down." For $\theta \in (\theta_\delta, \pi - \theta_\delta) \cup (\pi + \theta_\delta, -\theta_\delta)$ the probabilities coincide with the quantum probabilities for a spin measurement of a spin-1/2 particle, as we showed earlier for the model without shielding tubes. For $\delta = 0$, we have the following. The states which make an angle $\theta \in [0, \pi/2)$ with the measurement direction u will always give the outcome "spin up." If the state of the entity makes an angle $\theta \in (\pi/2, \pi]$ with the measurement direction u, then we always get the outcome "spin down." For $\theta = \pi/2$ the probability of every outcome equals 0.5, which again coincides with the probabilities for a spin measurement of a quantum spin-1/2 particle. So we have the situation that for $\delta \in [0, 1]$ all the states for which the projection point lies between the two tubes have probabilities of the respective outcomes equal to those of a quantum spin-1/2 measurement.

Let us now apply the geometric interpretation procedure considered before for a pure quantum case to measurements with $\delta \neq 1$. We place two parallel planes in the cylinder orthogonal on its ground plane and orthogonal to the $[-u, u]$ axis in the way described on Fig. 4.

We now define the membership function of a fuzzy set that represents a property a = *particle has "spin up"* as follows. For states in the interval $[-\theta_\delta, \theta_\delta]$ the membership function follows the top of the cylinder (it has value 1). For states between the two parallel planes the height curve follows the intersection ellipse. For the states in the interval $[\pi - \theta_\delta, \pi + \theta_\delta]$ the membership function follows the bottom of the cylinder (it has height zero). Again it is easy to calculate that if we unfold the surface of the cylinder, we get the value of the membership function at each point that equals the probability of obtaining the outcome "spin up" in a spin measurement. In other words, the membership function of a so-defined fuzzy subset of the set of points that represent pure states can be seen as the function which defines the probability with which the entity in that initial state will yield

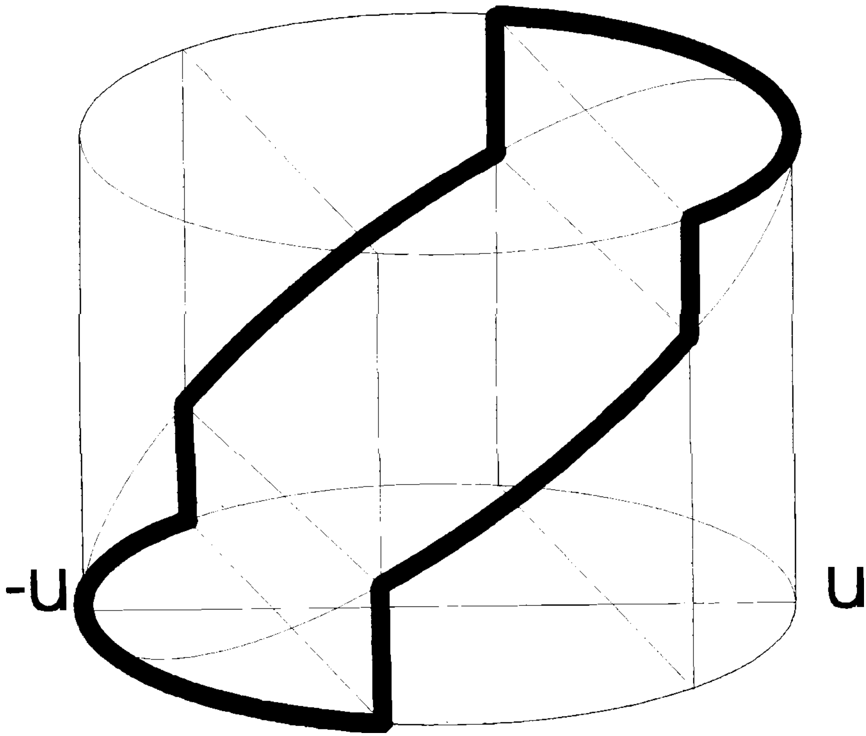

Fig. 4. Geometrical interpretation of the transition probabilities for the model with shielding of the elastic ($\delta \neq 1$).

the "spin up" outcome in the measurement. Therefore, we can assign to this fuzzy set the property "spin is up." For the states lying in the interval $[-\theta_\delta, \theta_\delta]$ the measurement will yield with certainty the outcome "spin is up." For the states lying in the interval $[\pi - \theta_\delta, \pi + \theta_\delta]$ the measurement will yield with certainty the outcome "spin is down." For the other states only a probability

$$\cos^2\left(\frac{\theta}{2}\right) \in \left(\cos^2\left(\frac{\pi - \theta_\delta}{2}\right), \cos^2\left(\frac{\theta_\delta}{2}\right)\right) = \left(\frac{1-\delta}{2}, \frac{1+\delta}{2}\right)$$

which coincides with the respective quantum mechanical probabilities, is given. In the classical limit ($\delta \to 0$) every membership function obtained in such a way induces a quasicrisp set, because it takes values 1 and 0, except on a zero measure subset of the state space.

The physical meaning of the above-described procedure is that we could consider the measurements with $\delta \neq 1$ as "degenerate" spin measurements in the sense that in some region around the "spin up" zone it always gives the outcome "spin up," and likewise for the "spin down" outcome. Outside these "degenerate" regions the measurements show no deviation from the usual quantum behavior and therefore the same probabilities as for an ideal quantum spin measurement are found. Let us notice that if a spin-measuring device yielding the outcomes described above is ever realized in practice, it would not allow a description in terms of a two-dimensional complex Hilbert space and would force us to go beyond orthodox quantum theory:

Theorem 2. The δ-model can be represented in a complex Hilbert space iff $\delta = 1$.

Proof. Let us suppose that the model can be represented in a complex Hilbert space. This means that every state p of the model can be represented by a unit vector x, such that the state is the ray generated by this unit vector. It also means that the transition probability $P(p|q)$ between two states p and q is given by $|\langle x, y\rangle|^2$, where x and y are the unit vectors representing the states p and q. Suppose now that $\delta < 1$. Let us call p the eigenstate of the "upper" θ_δ-spherical sector $[-\theta_\delta, \theta_\delta]$ where the transition probabilities to the state p of the states contained in this spherical sector are 1, and let p be represented by the unit vector x of the Hilbert space. Since $\delta < 1$ and therefore $\theta_\delta \neq 0$, we can choose two states q and r, represented in the Hilbert space by two unit vectors y and z, on the border of this spherical sector, such that $P(q|p) = 1$ and $P(r|p) = 1$, but such that for a fixed δ we have $P(r|q) < 1$. This means that $|\langle y, x\rangle| = |\langle z, x\rangle| = 1$, which implies that $y = x = z$, but $|\langle y, z\rangle| < 1$, which implies that $z \neq y$. From this we can conclude that δ has to be equal to 1 if the model can be represented in a Hilbert space.

For $\delta = 1$ the particle has the same probabilities as a quantum spin-1/2 particle for all the states, and for $\delta = 0$ it behaves deterministically, except on the two-point set of "unstable equilibrium," where we used the convention that 50% of the particles give outcome "spin up" and 50% of the particles give outcome "spin down." Therefore we could call $\delta = 0$ the deterministic limit of the entity, or even the classical limit of the entity, if we are only interested in the probabilistic features of the model.

4. FUZZY SET MODEL OF TWO-DIMENSIONAL HILBERTIAN QUANTUM LOGIC AND ITS DEFUZZYFICATION

In the previous section we showed that the probabilities appearing in the hidden measurement model can be given a geometrical interpretation which allows to represent a property of a spin-1/2 particle by a fuzzy subset of a section of a Poincaré sphere. This discussion for spin measurements of spin-1/2 particles can be repeated for experiments which measure the linear polarization of photons. The polarization of photons can be described in the same Hilbert space and, therefore, we can again use the representation of states of a system on the Poincaré sphere (Mielnik, 1968). Now, the interpretation of points on the sphere is as follows: If we consider a measurement of the polarization, then the north and south poles of the sphere correspond to opposite circular polarizations. The points on the equator represent linearly polarized states and the other points on the surface correspond to elliptically polarized states. Two opposite points on the sphere represent two opposite polarizations. Pure states correspond to points on the surface of the sphere; the interior points correspond to mixed states, which we do not have to consider separately since they can always be decomposed into their pure components. Let us consider linearly polarized photons. If a beam of such photons passes a linear polarizer P with certainty, we can say that photons in this beam are polarized in that direction or that they possess (to the extent 100%) the property of *passing through the polarizer P*. If we then place another linear polarizer P_θ parallel to the first one in the beam such that the polarizer P makes an angle θ with it, then the probability that a photon passes the second polarizer is given by $\cos^2 \theta$. In particular, two polarizations are opposite if they make an angle $\pi/2$, while on the sphere two polarization states are opposite if they make an angle π. Thus, the membership function of a fuzzy set that describes the property of *passing through a polarizer* P_θ is the same as the membership function of a fuzzy set that represents a property of *obtaining the outcome "spin up"* for a spin-1/2 particle during a quantum spin measurement with an argument $\theta/2$ (giving the spin state of a spin-1/2 particle) replaced by θ [giving the polarization state of the (beam of) photon(s)].

So we can describe the properties of an entity described in two-dimensional Hilbert space by fuzzy sets. We showed in Section 3 that further we can apply to these sets (at least in the case of fuzzy sets obtained in the "parametrized hidden measurements model") a suitable "defuzzyfication" procedure and, by letting the parameter δ converge to zero, we can make a "continuous" transition to (quasi-)crisp sets which describe properties of classical systems. However, in Section 3 we were concerned with a single set which describes a single property of a physical system, while now we shall investigate the structure of the whole family of such sets that describe all properties of a physical system. In particular, we are interested in whether for any $\delta \in [0, 1]$ the family of all such sets is a quantum logic in the sense described in Theorem 1, i.e., whether it fulfills the conditions (a)–(d) of this theorem.

The answer to the above-posed problem is positive and we can show it as follows:

In the pure quantum case ($\delta = 1$) the family of all fuzzy sets that represent properties of the system consists of the following:

1. Two crisp sets: the empty set, which represents a property that is never true (e.g., "the system does not exist"), and the whole set of states, which represents a property that is always true (e.g., "the system exists").

2. All fuzzy sets obtained in the way described in Section 3, i.e., obtained by intersecting a cylinder of a height 1 by planes which meet the top and the bottom of the cylinder at two opposite points (left-hand side of Fig. 2). If we unfold the surface of the cylinder so that the ground circle (which consists of points that represent pure states of a system) becomes an interval of length 2π, all possible intersection curves (right-hand side of Fig. 2) are of the form

$$f(\theta) = \cos^2 \frac{\theta - \varphi}{2} \tag{11}$$

with a parameter φ taking all values in the interval $[0, 2\pi)$.

Now we can easily see that all conditions of Theorem 1 that force the so-defined family of sets to be a quantum logic are fulfilled: Condition (a) is obvious. Condition (b) follows from the fact that if a membership function of a fuzzy set $\mathcal{A}$ intersects the surface of a cylinder along the curve $\cos^2[(\theta - \varphi)/2]$ then its fuzzy complement does so along the curve $\cos^2[(\theta - \varphi - \pi)/2]$ (subtraction modulo 2π) i.e., it also belongs to the considered family of fuzzy sets. Conditions (c) and (d) are fulfilled in a trivial way since the only pairs of weakly disjoint sets in the considered family are, besides of pairs that consist of a set and its complement, those pairs that contain the empty set. Indeed, for any two curves $\cos^2[(\theta - \varphi_1)/2]$ and $\cos^2[(\theta - \varphi_2)/2]$ there necessarily exist points in the interval $[0, 2\pi[$ such that $\cos^2[(x -$

$\varphi_1)/2] + \cos^2[(x - \varphi_2)/2] > 1$, unless $\varphi_2 = \varphi_1 + \pi$ (modulo 2π), in which case these two curves represent two complementary fuzzy sets.

Since two different curves of the form (11) necessarily intersect at two points, the order-theoretic structure of the considered family of sets is identical to the structure of a lattice of subspaces of a two-dimensional vector space. The Hasse diagram of such a lattice consists of a zero-dimensional subspace (in our case, the empty set) as the least element *0* of the uncountable family of one-dimensional subspaces [in our case fuzzy sets defined by the intersection curves (11)] none of which precedes any other (p, p', q, q', r, r', . . . in the figure), and of the whole vector space (in our case the set of all states) as the greatest element *I* (Fig. 5).

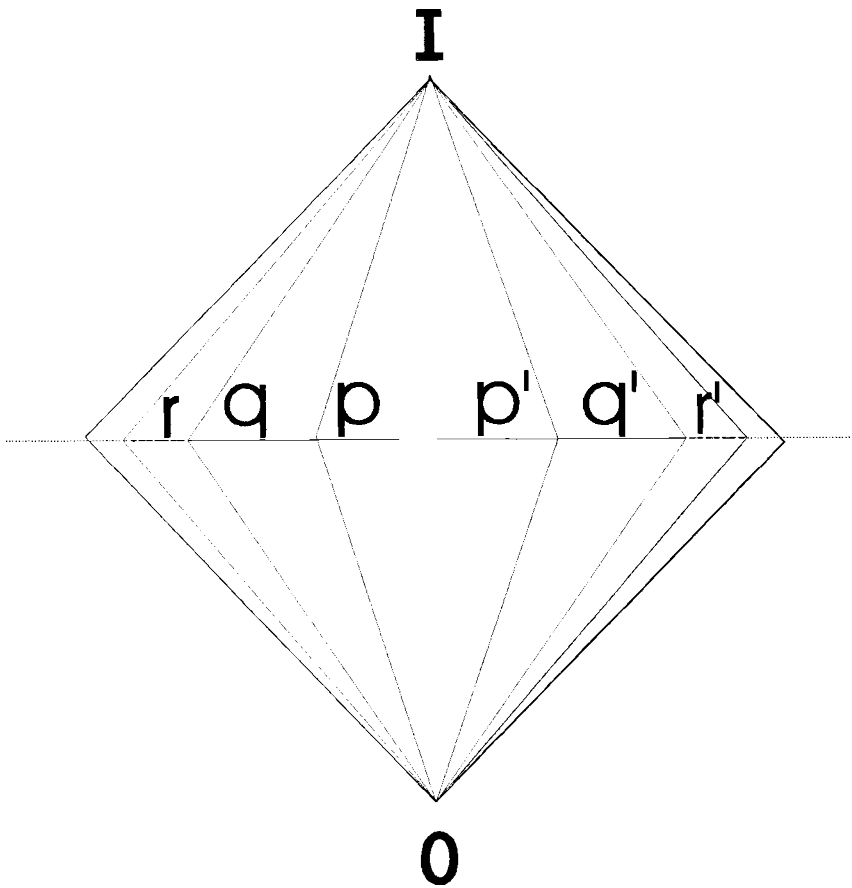

Fig. 5. The Hasse diagram of the lattice of subspaces of a two-dimensional vector space. Only a few of the uncountable family of one-dimensional subspaces are displayed here.

Let us now pass to the intermediate cases when $\delta \in (0, 1)$. In order to make the analysis easier we shall restrict attention to the intersection curves defined on the interval $[0, 2\pi]$ that are obtained by unfolding the surface of a cylinder (see Fig. 4) since they define respective fuzzy sets in an unambiguous way. In all these cases the families of curves that we are interested in contain, beside two constant functions (the null function and the unit function), all curves of the form

$$f(\theta) = \begin{cases} 1, & \theta \in [\varphi - \theta_\delta, \varphi + \theta_\delta] \\ \cos^2 \dfrac{\theta - \varphi}{2}, & \theta \in (\varphi + \theta_\delta, \varphi - \theta_\delta + \pi) \cup (\varphi + \theta_\delta + \pi, \varphi - \theta_\delta) \\ 0, & \theta \in [\varphi - \theta_\delta + \pi, \varphi + \theta_\delta + \pi] \end{cases} \tag{12}$$

where $\varphi \in [0, 2\pi)$, $\delta \in (0, 1)$, $\theta_\delta = \arccos \delta$, and all additions and subtractions are modulo 2π. It is easy to see that all arguments used previously to show that in the pure quantum case (which, actually, means that $\delta = 1$) the family of all sets that represent properties of the physical system is a quantum logic remain valid also for any $\delta \in (0, 1)$.

Finally, let us note that "quasicrisp" sets obtained in the deterministic limit ($\delta = 0$) are of the form

$$f(\theta) = \begin{cases} 1, & \theta \in (\varphi - \pi/2, \varphi + \pi/2) \\ \frac{1}{2}, & \theta \in \{\varphi - \pi/2, \varphi + \pi/2\} \\ 0 & \theta \in (\varphi + \pi/2, \varphi + \frac{3}{2}\pi) \end{cases} \tag{13}$$

(additions and subtractions modulo 2π). The value $f(\theta) = 1/2$ for $\theta = \varphi - \pi/2$ and $\theta = \varphi + \pi/2$ follows from our previous assumption that this case represents an "unstable equilibrium" for which 50% of the measurements yield the result "spin up" and the other 50% yield the result "spin down." Again, as in the previous cases it is easy to see that the conditions (a)–(d) of Theorem 1 are fulfilled [conditions (c) and (d) again in a trivial way], so the obtained family of fuzzy sets which represent properties of the physical system is a quantum logic.

Let us note that the problem with the points of "unstable equilibrium" is, in a sense, unphysical, since these points are of measure zero. Therefore, from the point of view of an experimentalist, we can equally well adopt the convention that, e.g.,

$$f(\theta) = \begin{cases} 1, & \theta \in [\varphi - \pi/2, \varphi + \pi/2) \\ 0, & \theta \in [\varphi + \pi/2, \varphi + \frac{3}{2}\pi) \end{cases} \tag{14}$$

(additions and subtractions modulo 2π), which means that now all sets that represent properties of the physical system are crisp sets. It can be easily checked that the family of all crisp sets determined by (14) form, together with the empty set and the set of all states, a so called *concrete logic* (see, e.g., Pták and Pulmannová, 1991, p. 2), which is a collection Δ of crisp subsets of a set Ω that satisfies the following conditions:

(1) $\emptyset \in \Delta$.
(2) If $A \in \Delta$, then $\Omega \setminus A \in \Delta$.
(3) If $\{A_i: i \in N\} \subset \Delta$ is a countable family of mutually disjoint subsets in Ω, the $\cup_{i \in N} A_i \in \Delta$.

Again the last condition is in our case trivially fulfilled since the only disjoint pairs of sets are, beside the pairs containing a set and its complement, those pairs that contain the empty set.

Since it is generally agreed that logics of classical physical systems are Boolean algebras, it is very important to check whether our logic obtained in the "deterministic limit" (i.e., when $\delta = 0$) is a Boolean algebra. To make the analysis easier, let us study the concrete logic that contains the empty set, the set of all states, and all (crisp) sets determined by the formula (14) for $\varphi \in [0, 2\pi)$. The problem is solved when we make use of two facts quoted by Pták and Pulmannová (1991):

Fact 1. In a concrete logic Δ two elements $A, B \in \Delta$ are compatible if and only if $A \cap B \in \Delta$.

Fact 2. A logic is a Boolean σ-algebra if and only if every pair of its elements is compatible.

Now it is obvious that our concrete logic obtained in the deterministic limit is not a Boolean algebra since it does not contain the intersection of any pair of its sets, besides the pairs consisting of a set and its complement and pairs containing the empty set or the set of all states. Therefore, it is better to call the limit $\delta \to 0$ the "deterministic limit" rather than the "classical limit."

This above-quoted result is, in fact, much expected: if two subsets A, B of a set of states represent two properties a, b of a physical system [in the sense that A (resp. B) consists of these states of a system for which a property a (resp. b), holds *when it is measured*], then the intersection $A \cap B$ consists of those states for which both these properties hold when measured *simultaneously*. However, in the hidden measurement approach we cannot simultaneously measure a spin of a particle in the direction $[u, -u]$ and in another direction $[v, -v]$, even for $\delta = 0$, which prevents the lattice of properties from being a Boolean algebra and makes it a "quantum spin-1/2 lattice" even in the deterministic limit. Therefore, we see that a "classical" character of a physical system is forced rather by simultaneous measurability of all its properties than by the lack of randomness in these measurements.

5. CONCLUSION

In this paper the procedure of "defuzzyfication" was applied to fuzzy sets that describe properties of quantum spin-1/2 particles which are represented in a two-dimensional Hilbert space. It was shown that although in the deterministic limit crisp sets are obtained, this defuzzyfication procedure does not suffice to endow the quantum logic (modeled by a suitable family of fuzzy sets) with the structure of a Boolean algebra. It was shown that the lattice of properties (i.e., the quantum logic) remains a pure quantum one, even in the deterministic limit of the entity. The reason for this is that the hidden measurement approach in general does not allow the simultaneous measurement of two properties.

ACKNOWLEDGMENTS

The paper was written as a part of the joint Polish–Flemish Research Project "Probing the Structure of Quantum Mechanics: New Probability Models for New Experiments on Quantum Particles." J.P. was also supported by University of Gdańsk Research Grant BW/5100-5-0276-7. B.D'H. is Research Assistant of the Fund for Scientific Research-Flanders (Belgium) (F.W.O.).

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